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Short Communication

Comment on “A positivity preserving and well-balanced DG scheme using finite volume subcells in almost dry regions”

Longxiang Li, Qinghe Zhang*

State Key Laboratory of Hydraulic Engineering Simulation and Safety, Tianjin University, Tianjin 300072, China

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ABSTRACT

This comment concerns a mistake in the proving of positivity-preserving property for the well-balanced discontinuous Galerkin methods presented by Meister and Ortleb [Meister A, Ortleb S, A positivity preserving and well-balanced DG scheme using finite volume subcells in almost dry regions, Applied Mathematics and Computation 272 (2016) 259–273]. Furthermore, we provide a correct proof which gives the same result.

1. Introduction

Meister and Ortleb [1] recently proposed a positivity preserving and well-balanced discontinuous Galerkin (DG) methods with applying a subcell finite volume (FV) discretization in those almost dry cells. Through the elaborate design of decomposing the DG cell into a specific number of subcell FV elements, the shoreline of moving water boundary could be well captured and some important properties, such as mass conservation and well-balancing, are preserved. In order to preserve the positivity of water depths, the presented method in [1] employed the explicit time stepping method and provide a restricted time step size for the HLL numerical flux to avoid an excess amount of water flowing out of the elements, which can be seen as an extending work of Kurganov and Petrova [2] and Xing and Zhang [3]. However, we found that there is a mistake in proving the proof of positivity in DG scheme with order $N=0$. This error in derivation is corrected by using an alternative inequality where the same result can be obtained.

2. An error in the proof of positivity-preserving with DG scheme

In proving the positivity of water depths in [1], the evaluation of H_i^{n+1} with $N=0$ is written in two terms as

$$
\begin{aligned}
H_i^{n+1} &= \left[1 - \frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H_i^{*,n}}{H_i^n}\frac{\tilde{s}_{R,iK}v_i^n\cdot n_{iK} - \tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK} - \tilde{s}_{L,iK}}\right]H_i^n \\
&\quad + \left[\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H_{j(i,K)}^{*,n}}{H_{j(i,K)}^n}\frac{\tilde{s}_{L,iK}v_i^n\cdot n_{iK} - \tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK} - \tilde{s}_{L,iK}}\right]H_{j(i,K)}^n, \qquad (1)
\end{aligned}
$$

where

* Corresponding author.
E-mail address: qhzhang@tju.edu.cn (Q. Zhang).

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$$\tilde{s}_{L,iK} = min\{s_L, 0\} = min\Big\{min\Big\{v^- \cdot n - \sqrt{H^-}, v^+ \cdot n - \sqrt{H^+}\Big\},\ 0\Big\}$$
$$\tilde{s}_{R,iK} = max\{s_R, 0\} = max\Big\{max\Big\{v^- \cdot n + \sqrt{H^-}, v^+ \cdot n + \sqrt{H^+}\Big\},\ 0\Big\} \tag{2}$$

are two auxiliary speeds, which can assist to rewrite the HLL flux in a compact form. As $\sqrt{H^\pm} \geq 0$ is always true, the relations between the auxiliary speeds and normal velocities hold as

$$\tilde{s}_{L,iK} \leq v^- \cdot n,\ \tilde{s}_{L,iK} \leq v^+ \cdot n,$$
$$\tilde{s}_{R,iK} \geq v^- \cdot n,\ \tilde{s}_{R,iK} \geq v^+ \cdot n. \tag{3}$$

It is important to point out that these inequalities are key ingredients in the following proof. To ensure the non-negative of H_i^{n+1} in Eq. (1), a CFL condition is obtained from preserving the positivity of the first term, while the second term is reported to be positive in [1] with the following inequality as

$$\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\frac{\tilde{s}_{L,iK}v^n_{j(i,K)}\cdot n_{iK}-\tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK}-\tilde{s}_{L,iK}} \geq -\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\tilde{s}_{L,iK} \tag{4}$$

However, since $\tilde{s}_{L,iK} = \min\{s_L, 0\} \leq 0$, then $\tilde{s}_{L,iK}\mathbf{v}^n_{j(i,K)} \cdot \mathbf{n}_{iK} \leq \tilde{s}^2_{L,iK}$ and the correct inequality should be

$$\begin{aligned}&\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\frac{\tilde{s}_{L,iK}v^n_{j(i,K)}\cdot n_{iK}-\tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK}-\tilde{s}_{L,iK}}\\ &\quad\leq \frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\frac{\tilde{s}^2_{L,iK}-\tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK}-\tilde{s}_{L,iK}},\end{aligned} \tag{5}$$

i.e.

$$\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\frac{\tilde{s}_{L,iK}v^n_{j(i,K)}\cdot n_{iK}-\tilde{s}_{L,iK}\tilde{s}_{R,iK}}{\tilde{s}_{R,iK}-\tilde{s}_{L,iK}} \leq -\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}\tilde{s}_{L,iK}. \tag{6}$$

It is clear that Eq. (6) could not be used to prove the positivity of the second term.

3. Correction of the proof

Nevertheless, we could use an alternative inequality form from Eq. (3) and expressed as

$$\tilde{s}_{R,iK} - \tilde{s}_{L,iK} \geq \tilde{s}_{R,iK} - \mathbf{v}^n_{j(i,K)} \cdot \mathbf{n}_{iK} \geq 0, \tag{7}$$

then if $\tilde{s}_{R,iK} - \tilde{s}_{L,iK} \neq 0$, the following inequality is derived

$$1 \geq \frac{\tilde{s}_{R,iK} - \mathbf{v}^n_{j(i,K)} \cdot \mathbf{n}_{iK}}{\tilde{s}_{R,iK} - \tilde{s}_{L,iK}} \geq 0, \tag{8}$$

Rewrite the second term as

$$\frac{\Delta t}{|\tau_i|}\sum_{K=1}^{3}|\Gamma_{iK}|\frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}}(-\tilde{s}_{L,iK})\frac{\tilde{s}_{R,iK}-v^n_{j(i,K)}\cdot n_{iK}}{\tilde{s}_{R,iK}-\tilde{s}_{L,iK}} \tag{9}$$

and substitute the above inequality, then the positivity of this term can be easily proved with $-\tilde{s}_{L,iK} \geq 0$ and $1 \geq \frac{H^{*,n}_{j(i,K)}}{H^n_{j(i,K)}} \geq 0$, which denoting that the positivity of H_i^{n+1} is preserved.

4. Conclusion

In this comment, we show a mistake in proving the non-negative of H_i^{n+1} with $N=0$ and give the corrected proof with an alternative inequality.

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